Proof: The Shortest Path Between Two Points Is a Straight Line

A simple proof using coordinate rotation and the arc length formula

STATEMENT

The shortest path connecting two distinct points A and B in a Euclidean plane is the straight line segment joining them.

ASSUMPTIONS

1. Points A and B are distinct and lie in a 2D Euclidean plane
2. The path is continuous and differentiable
3. We use the standard arc length formula from calculus
4. Distance is measured using the Euclidean metric

PROOF

Step 1 — Setup

Place point A at the origin (0, 0). Rotate the coordinate axes so that point B lies on the positive x-axis at (d, 0), where d = |AB|.

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// Rotation only changes perspective — it doesn't change any distances.
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Step 2 — Define two paths

L_1 (straight line): the direct line from A to B, now lying on the x-axis, y = 0.

L_2 (arbitrary path): any smooth curve y = g(x) connecting A(0,0) to B(d,0).

Step 3 — Length of the straight line L_1

$$\text{Arc length formula: } L = \int \sqrt{(1 + (dy/dx)^2)}\, dx$$

For y = 0, dy/dx = 0, so:

$$l_1 = \int_0^d \sqrt{(1 + 0^2)}\, dx = \int_0^d 1\, dx = d$$

Step 4 — Length of the curved path L_2

For y = g(x):

$$l_2 = \int_0^d \sqrt{(1 + g'(x)^2)}\, dx$$

Step 5 — Key inequality

For any real number g'(x), squaring gives a non-negative result:

$$g'(x)^2 \geq 0$$

Add 1 to both sides, then take the square root (order is preserved):

$$\sqrt{(1 + g'(x)^2)} \geq \sqrt{1} = 1$$

// Any deviation from a flat (straight) path makes this quantity strictly larger.

Step 6 — Integrate both sides

$$\int_0^d \sqrt{(1 + g'(x)^2)}\, dx \geq \int_0^d 1\, dx$$

which gives:

$$l_2 \geq l_1$$

Step 7 — Equality condition

Equality $l_2 = l_1$ holds only when g'(x) = 0 for every x, which means g(x) is constant — i.e. the straight line itself.

Conclusion: $l_2 \geq l_1$ for every path, with equality only for the straight line.
Hence the straight line is the unique shortest path between two points. ∎

COMMENTS

Why rotate axes? It makes the straight path simply $y = 0$, so any curved path's "extra" length becomes visually and algebraically obvious.

The core idea: The inequality $\sqrt{(1 + g'(x)^2)} \geq 1$ says that moving away from a straight (horizontal) direction always costs extra distance.

No advanced tools needed: Only the standard arc length formula from calculus is used — no variational calculus or Euler-Lagrange equations.

Intuition: Walking 10 m east plus 5 m north covers $\sqrt{(10^2+5^2)} \approx 11.18$ m — more than 10 m. Any vertical detour adds distance.

Generalizes easily: The same argument works in 3D or any n-dimensional Euclidean space — just rotate axes so AB lies along one axis.